

On some properties of contracting matrices

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Abstract

The concepts of paracontracting, pseudocontracting and nonexpanding operators have been shown to be useful in proving convergence of asynchronous or parallel iteration algorithms. The purpose of this paper is to give characterizations of these operators when they are linear and finite-dimensional. First we show that pseudocontractivity of stochastic matrices with respect to $\|\cdot\|_\infty$ is equivalent to the scrambling property, a concept first introduced in the study of inhomogeneous Markov chains. This unifies results obtained independently using different approaches. Secondly, we generalize the concept of pseudocontractivity to set-contractivity which is a useful generalization with respect to the Euclidean norm. In particular, we demonstrate non-Hermitian matrices that are set-contractive for $\|\cdot\|_2$, but not pseudocontractive for $\|\cdot\|_2$ or $\|\cdot\|_\infty$. For constant row sum matrices we characterize set-contractivity using matrix norms and matrix graphs. Furthermore, we prove convergence results in compositions of set-contractive operators and illustrate the differences between set-contractivity in different norms. Finally, we give an application to the global synchronization in coupled map lattices.

Key words: coupled map lattice, Markov chains, nonexpanding operators, paracontractive operators, pseudocontractive operators, scrambling matrices, stochastic matrices, synchronization.

1 Introduction

Definition 1 ([1]) *Let $\|\cdot\|$ be a vector norm in \mathbb{C}^n . An n by n matrix B is nonexpansive with respect to $\|\cdot\|$ if*

$$\forall x \in \mathbb{C}^n, \|Bx\| \leq \|x\| \quad (1)$$

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B is called paracontracting with respect to $\|\cdot\|$ if

$$\forall x \in \mathbb{C}^n, Bx \neq x \Leftrightarrow \|Bx\| < \|x\| \quad (2)$$

It is easy to see that normal matrices with eigenvalues in the unit circle and for which 1 is the only eigenvalue of unit norm is paracontractive with respect to $\|\cdot\|_2$.

Definition 2 For a vector $x \in \mathbb{C}^n$ and a closed set X^* , y^* is called a projection vector of x onto X^* if $y^* \in X^*$ and

$$\|x - y^*\| = \min_{y \in X^*} \|x - y\|$$

The distance of x to X^* is defined as $d(x, X^*) = \|x - P(x)\|$ where $P(x)$ is a projection vector of x onto X^* .

Even though the projection vector is not necessarily unique, we write $P(x)$ when it is clear which projection vector we mean or when the choice is immaterial. Let us denote $e = (1, \dots, 1)^T$. The proof of the following Lemma is relatively straightforward and thus omitted.

Lemma 1 If $x \in \mathbb{R}^n$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$, the projection vector $P(x)$ of x onto X^* is αe where:

- for the norm $\|\cdot\|_2$, $\alpha = \frac{1}{n} \sum_i x_i$ and $d(x, X^*) = \sqrt{\sum_i (x_i - \alpha)^2}$.
- for the norm $\|\cdot\|_\infty$, $\alpha = \frac{1}{2} (\max_i x_i + \min_i x_i)$, and $d(x, X^*) = \frac{1}{2} (\max_i x_i - \min_i x_i)$.
- for the norm $\|\cdot\|_1$, $d(x, X^*) = \sum_{i=\lceil \frac{n}{2} \rceil + 1}^n \hat{x}_i - \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \hat{x}_i$ and
 - for n odd, $\alpha = \hat{x}_{\lceil \frac{n}{2} \rceil}$.
 - for n even, α can be chosen to be any number in the interval $[\hat{x}_{\frac{n}{2}}, \hat{x}_{\frac{n}{2}+1}]$.

Here \hat{x}_i are the values x_i rearranged in nondecreasing order $\hat{x}_1 \leq \hat{x}_2 \leq \dots$.

The property of paracontractivity is used to show convergence of infinite products of paracontractive matrices and this in turn is used to prove convergence in various parallel and asynchronous iteration methods [2]. In [3] this property is generalized to pseudocontractivity.

Definition 3 ([3]) Let T be an operator on \mathbb{R}^n . T is nonexpansive with respect to $\|\cdot\|$ and a closed set X^* if

$$\forall x \in \mathbb{R}^n, x^* \in X^*, \|Tx - x^*\| \leq \|x - x^*\| \quad (3)$$

T is pseudocontractive with respect to $\|\cdot\|$ and X^* if it is nonexpansive with respect to $\|\cdot\|$ and X^* and

$$\forall x \notin X^*, d(Tx, X^*) < d(x, X^*) \quad (4)$$

Ref. [3] shows that there are pseudocontractive nonnegative matrices which are not paracontractive with respect to $\|\cdot\|_\infty$ and proves a result on the convergence of infinite products of pseudocontractive matrices. Furthermore, Ref. [3] studies a class of matrices for which a finite product of matrices from this class of length at least $n - 1$ is pseudocontractive in $\|\cdot\|_\infty$.

The purpose of this paper is multifold. First we show that for stochastic matrices with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$, pseudocontractive matrices are equivalent to scrambling matrices and thus are simply characterized. The concept of scrambling matrices is first introduced in the study of weak ergodicity in inhomogeneous Markov chains and this equivalence allows us to unify several results obtained independently using these different concepts.

The second goal of this paper is to generalize pseudocontractivity by introducing the concept of set-contractivity. We prove a convergence result of set-contractive matrices and show existence of set-contractive matrices in $\|\cdot\|_2$ that are not pseudocontractive with respect to $\|\cdot\|_2$ or $\|\cdot\|_\infty$. We study set-contraction with respect to $\|\cdot\|_2$ in terms of matrix norms and graphs of matrices.

Finally, we apply these results to the global synchronization of coupled map lattices.

We concentrate on the case where T are matrices and X^* is the span of the corresponding Perron eigenvector. If the Perron eigenvector is strictly positive, then as in [3], a scaling operation $T \rightarrow W^{-1}TW$ where W is the diagonal matrix with the Perron eigenvector on the diagonal, transforms T into a matrix for which the Perron eigenvector is e . Therefore in the sequel we will focus on constant row sum matrices with $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$.

2 Pseudocontractivity and scrambling stochastic matrices

Scrambling matrices were first defined in [4] to study weak ergodicity of inhomogeneous Markov chains.

Definition 4 *A matrix A is scrambling if for any pair of indices i, j , there exists k such that $A_{ik} \neq 0$ and $A_{jk} \neq 0$.*

Definition 5 *For a real matrix A , $\mu(A)$ is defined as*

$$\mu(A) = \min_{j,k} \sum_i \min(A_{ji}, A_{ki})$$

For nonnegative matrices with row sums $\leq r$, it is clear that $0 \leq \mu(A) \leq r$ with $\mu(A) > 0$ if and only if A is scrambling.

Definition 6 For a real matrix A , define $\delta(A) \geq 0$ as

$$\delta(A) = \max_{i,j} \sum_k \max(0, A_{ik} - A_{jk}) \geq \max_{i,j,k} (A_{ik} - A_{jk})$$

If A has constant row sums, then $\delta(A) = \frac{1}{2} \max_{i,j} \sum_k |A_{ik} - A_{jk}|$.

Theorem 1 If A is a matrix where each row sum is equal to or less than r , then $\delta(A) \leq r - \mu(A)$.

Proof: Ref. [5] proved this for the case of stochastic matrices and the same proof applies here. \square

Theorem 2 If A is a real matrix with constant row sums and $x \in \mathbb{R}^n$, then $\max_i y_i - \min_i y_i \leq \delta(A) (\max_i x_i - \min_i x_i)$ where $y = Ax$.

Proof: The proof is similar to the argument in [5]. Let $x_{\max} = \max_i x_i$, $x_{\min} = \min_i x_i$, $y_{\max} = \max_i y_i$, $y_{\min} = \min_i y_i$.

$$\begin{aligned} y_{\max} - y_{\min} &= \max_{i,j} \sum_k (A_{ik} - A_{jk}) x_k \\ &\leq \max_{i,j} (\sum_k \max(0, A_{ik} - A_{jk}) x_{\max} + \sum_k \min(0, A_{ik} - A_{jk}) x_{\min}) \end{aligned} \quad (5)$$

Since A has constant row sums, $\sum_k A_{ik} - A_{jk} = 0$, i.e.

$$\sum_k \max(0, A_{ik} - A_{jk}) + \sum_k \min(0, A_{ik} - A_{jk}) = 0$$

This means that

$$\begin{aligned} y_{\max} - y_{\min} &\leq \max_{i,j} (\sum_k \max(0, A_{ik} - A_{jk})) (x_{\max} - x_{\min}) \\ &\leq \delta(A) (x_{\max} - x_{\min}) \end{aligned} \quad (6)$$

\square

The following result shows that pseudocontractivity of stochastic matrices with respect to $\|\cdot\|_{\infty}$ is equivalent to the scrambling condition and thus can be easily determined.

Theorem 3 Let A be a stochastic matrix. The matrix A is pseudocontractive with respect to $\|\cdot\|_{\infty}$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}^n\}$ if and only if A is a scrambling matrix.

Proof: Let $x^* \in X^*$. Then $Ax^* = x^*$ and thus $\|Ax - x^*\|_{\infty} = \|A(x - x^*)\|_{\infty} \leq \|x - x^*\|_{\infty}$. Thus all stochastic matrices are nonexpansive with respect to $\|\cdot\|_{\infty}$.

and X^* . Suppose A is a scrambling matrix. Then $\mu(A) > 0$, and $\delta(A) < 1$ by Theorem 1. By Lemma 1 and Theorem 2, A is pseudocontractive. Suppose A is not a scrambling matrix. Then there exists i, j such that for each k , either $A_{ik} = 0$ or $A_{jk} = 0$. Define x as $x_k = 1$ if $A_{ik} > 0$ and $x_k = 0$ otherwise. Since A is stochastic, it does not have zero rows and thus there exists k' and k'' such that $A_{ik'} = 0$ and $A_{jk''} > 0$. This means that $x \notin X^*$. Let $y = Ax$. Then $y_i = 1$ and $y_j = 0$. This means that $\max_i y_i - \min_i y_i \geq 1 = \max_i x_i - \min_i x_i$, i.e. A is not pseudocontractive. \square

With Theorem 3 several results which were shown independently can now be seen to be equivalent. For instance, in [6] it was shown that for stochastic matrices with positive diagonal entries and whose interaction digraph¹ contains a spanning directed tree a finite product of $n - 1$ or more such matrices is scrambling. In [7] it was shown that such matrices are irreducible or 1-reducible² and this result in [6] then mirrors Proposition 3.3 in [3].

In [8] the convergence of a class of asynchronous iteration algorithms was shown by appealing to results about scrambling matrices. In [3] this result is proved using the framework of pseudocontractions. Theorem 3 shows that these two approaches are essentially the same.

3 Set-nonexpansive and set-contractive operators

Consider the stochastic matrix

$$A = \begin{pmatrix} 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \end{pmatrix}$$

¹ The directed graph of a square matrix A is defined as the graph with an edge from vertex i to vertex j if and only if $A_{ij} \neq 0$. The interaction digraph of a matrix A is obtained from the directed graph of A by reversing the orientation of all the edges, i.e. it is the graph of A^T .

² A matrix is 1-reducible if after simultaneous row and column permutation it can

be written in the form $\begin{pmatrix} B_{11} & B_{12} & \cdots \\ & B_{22} & B_{23} & \cdots \\ & & \ddots & \\ & & & B_{kk} \end{pmatrix}$ such that B_{ii} are irreducible and for

each $i < k$, there exists $j > i$ such that $B_{ij} \neq 0$.

The matrix A is not pseudocontractive with respect to the Euclidean norm $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since $\|A\|_2 = 1.088 > 1$. On the other hand, A satisfies Eq. (4)³. This motivates us to define the following generalization of pseudocontractivity:

Definition 7 *Let X^* be a closed set in \mathbb{R}^n . An operator T on \mathbb{R}^n is set-nonexpansive with respect to $\|\cdot\|$ and X^* if*

$$\forall x \in \mathbb{R}^n, d(Tx, X^*) \leq d(x, X^*)$$

An operator T on \mathbb{R}^n is set-contractive with respect to $\|\cdot\|$ and X^ if it is set-nonexpansive with respect to $\|\cdot\|$ and X^* and*

$$\forall x \notin X^*, d(Tx, X^*) < d(x, X^*).$$

The set-contractivity of an operator T is defined as

$$c(T) = \sup_{x \notin X^*} \frac{d(Tx, X^*)}{d(x, X^*)} \geq 0$$

There is a dynamical interpretation to Definition 7. If we consider the operator T as a discrete-time dynamical system, then T being set-nonexpansive and set-contractive imply that X^* is a globally nonrepelling invariant set and a globally attracting set of the dynamical system respectively [9].

Lemma 2 *T is set-nonexpansive with respect to $\|\cdot\|$ and X^* if and only if $T(X^*) \subseteq X^*$ and $c(T) \leq 1$. If T is set-contractive with respect to $\|\cdot\|$ and X^* , then the fixed points of T is a subset of X^* . If $T_1(X^*) \subseteq X^*$, then $c(T_1 \circ T_2) \leq c(T_1)c(T_2)$.*

Proof: The first statement is true by definition. The proof of the second statement is the same as in Proposition 2.1 in [3]. Suppose $T_1(X^*) \subseteq X^*$. Let $x \notin X^*$. If $T_2(x) \in X^*$, then $d(T_1 \circ T_2(x), X^*) = 0$. If $T_2(x) \notin X^*$, then $d(T_1 \circ T_2(x)) \leq c(T_1)d(T_2(x), X^*) \leq c(T_1)c(T_2)d(x, X^*)$. \square

Lemma 3 *Let X^* be a closed set such that $\alpha X^* \subseteq X^*$ for all $\alpha \in \mathbb{R}$. If T is linear and $T(X^*) \subseteq X^*$, then $c(T) = \sup_{\|x\|=1, P(x)=0} d(T(x), X^*)$.*

Proof: Let $\epsilon = \sup_{\|x\|=1, P(x)=0} d(T(x), X^*)$. Clearly $\epsilon \leq c(T)$. For $x \notin X^*$, 0 is a projection vector of $x - P(x)$. Since $T(P(x)) \in X^*$, this implies that $d(T(x), X^*) = d(T(x - P(x)), X^*) \leq \epsilon\|x - P(x)\| = \epsilon d(x, X^*)$, i.e. $\epsilon \geq c(T)$. \square

Lemma 4 *Let X^* be a closed set such that $\alpha X^* \subseteq X^*$ for all $\alpha \in \mathbb{R}$. An*

³ This can be shown using Theorem 6.

set-nonexpansive matrix T is set-contractive with respect to X^* if and only if $c(T) < 1$.

Proof: One direction is clear. Suppose T is set-contractive. By compactness

$$\sup_{\|x\|=1, P(x)=0} d(T(x), X^*) = \epsilon < 1$$

and the conclusion follows from Lemma 2 and Lemma 3. \square

If T is nonexpansive with respect to $\|\cdot\|$ and X^* , then

$$\|Tx - P(Tx)\| \leq \|Tx - P(x)\| \leq \|x - P(x)\|$$

and T is set-nonexpansive. Thus set-contractivity is more general than pseudo-contractivity. However, they are equivalent for stochastic matrices with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$.

Lemma 5 *With respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$, a stochastic matrix T is pseudocontractive if and only if it is set-contractive.*

Proof: Follows from the fact that a stochastic matrix is nonexpansive with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$. \square

Definition 8 ([10]) *A vector norm $\|\cdot\|$ on \mathbb{R}^n is monotone if*

$$\|(x_1, \dots, x_n)^T\| \leq \|(y_1, \dots, y_n)^T\|$$

for all x_i and y_i such that $|x_i| \leq |y_i|$. A vector norm $\|\cdot\|$ on \mathbb{R}^n is weakly monotone if

$$\|(x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_n)^T\| \leq \|(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n)^T\|$$

for all x_i and k .

The next result gives a necessary condition of set-contractivity of a matrix in terms of its graph.

Theorem 4 *Let A be a constant row sum matrix with row sums r such that $|r| \geq 1$. If A is set-contractive with respect to a weakly monotone vector norm $\|\cdot\|$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$, then the interaction digraph of A contains a spanning directed tree.*

Proof: If the interaction digraph A does not have a spanning directed tree, it was shown in [7] that after simultaneous row and column permutation, A can

be written as a block upper triangular matrix:

$$A = \begin{pmatrix} * & * & * & * \\ & \ddots & * & * \\ & & A_1 & 0 \\ & & & A_2 \end{pmatrix}$$

where $*$ are arbitrary entries and A_1 and A_2 are m_1 by m_1 and m_2 by m_2 square irreducible matrices respectively. Define $x = (0, \dots, 0, -a_1 e_1, a_2 e_2)^T \notin X^*$, where e_1 and e_2 are vectors of all 1's of length m_1 and m_2 respectively. Let $z = (0, \dots, 0, e_3)^T$ where e_3 is the vector of all 1's of length $m_1 + m_2$ and $Z^* = \{\alpha z : \alpha \in \mathbb{R}\}$. Note that the set of projection vectors of a fixed vector x to Z^* is a convex connected set. Let αz be a projection vector of x to Z^* . Suppose that for $a_1 = a_2 \neq 0$, $\alpha \neq 0$. Since $-\alpha z$ is a projection vector of $-x$ to Z^* and α (or at least a choice of α) depends continuously on a_1 and a_2 , by varying a_1 to $-a_1$ and varying a_2 to $-a_2$, α changes to $-\alpha$. This means that we can find a_1 and a_2 not both zero, such that 0 is a projection vector of x to Z^* . In this case $x \notin X^*$ and by weak monotonicity $d(x, Z^*) = d(x, X^*) = \|x\|$. It is clear that $y = Ax$ can be written as

$$y = \begin{pmatrix} * \\ \vdots \\ * \\ -ra_1 e_1 \\ ra_2 e_2 \end{pmatrix}$$

Let βe be a projection vector of y onto X^* . By the weak monotonicity of the norm,

$$d(y, X^*) = \|y - \beta e\| \geq \left\| \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-ra_1 - \beta)e_1 \\ (ra_2 - \beta)e_2 \end{pmatrix} \right\| = r \left\| x - \frac{\beta}{r} z \right\|$$

Since 0 is a projection vector of x onto Z^*

$$d(y, X^*) \geq |r|d(x, Z^*) \geq d(x, X^*)$$

Thus A is not set-contractive. □

3.1 max-norm

Theorem 5 *Let A be a matrix with constant row sum r . Then $c(A) = r - \mu(A)$ with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$. In particular, the matrix A is set-nonexpanding with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ if and only if $r - \mu(A) \leq 1$. The matrix A is set-contractive with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ if and only if $r - \mu(A) < 1$.*

Proof: $c(A) \leq r - \mu(A)$ follows from Lemma 1, Theorem 1 and Theorem 2. Since $c(A) \geq 0$, $c(A) = r - \mu(A)$ if $r - \mu(A) = 0$. Therefore we assume that $r - \mu(A) > 0$. Let j and k be such that $\mu(A) = \sum_i \min(A_{ji}, A_{ki})$. Define x such that $x_i = 1$ if $A_{ji} < A_{ki}$ and $x_i = 0$ otherwise. Since $r - \mu(A) > 0$, x is not all 0's or all 1's, i.e. $x \notin X^*$. Let $y = Ax$. Then by Lemma 1

$$\begin{aligned} 2d(y, X^*) &\geq y_k - y_j = \sum_{i, A_{ji} < A_{ki}} A_{ki} - A_{ji} \\ &= \sum_i A_{ki} - \sum_{i, A_{ji} \geq A_{ki}} A_{ki} - \sum_{i, A_{ji} < A_{ki}} A_{ji} \\ &= r - \mu(A) \end{aligned}$$

Since $2d(x, X^*) = 1$, it follows that $c(A) \geq r - \mu(A)$. \square

3.2 Euclidean norm

The following result characterizes set-contractivity of matrices with respect to $\|\cdot\|_2$ in terms of matrix norms.

Theorem 6 *Let A be an n by n constant row sum matrix and K be an n by $n-1$ matrix whose columns form a orthonormal basis of e^\perp . Then $c(A) = \|AK\|_2$ with respect to $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$. In particular $\|AK\|_2 \leq 1$ if and only if A is set-nonexpanding with respect to $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$. Similarly, $\|AK\|_2 < 1$ if and only if A is set-contracting with respect to $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$.*

Proof: Define $J = ee^T$ as the n by n matrix of all 1's. Note that $\|x\|_2 = \|Kx\|_2$ and $JK = 0$. Let $B = A - \frac{1}{n}J$. Then

$$\|AK\|_2 = \|BK\|_2 = \max_{\|x\|_2=1} \|BKx\|_2 = \max_{\|Kx\|_2=1} \|BKx\|_2 = \max_{x \perp e, \|x\|_2=1} \|Bx\|_2$$

By Lemma 1 $P(x) = \frac{1}{n}Jx$ and $d(Ax, X^*) = \|Bx\|_2$. Since A has constant row sums, $A(X^*) \subseteq X^*$ and by Lemma 3 $c(A) = \max_{P(x)=0, \|x\|_2=1} d(Ax, X^*) = \max_{P(x)=0, \|x\|_2=1} \|Bx\|_2$. Since $P(x) = 0$ if and only if $x \perp e$, this means that $c(A) = \|AK\|_2$. \square

3.3 weighted Euclidean norm

Definition 9 Given a positive vector w , the weighted 2-norm $\|\cdot\|_w$ is defined as

$$\|x\|_w = \sqrt{\sum_i w_i x_i^2}$$

Theorem 7 Let A be an n by n constant row sum matrix and K be as defined in Theorem 6. Let w be a positive vector such that $\max_i w_i = 1$ and $W = \text{diag}(w)$. Then $c(A) \leq \|W^{\frac{1}{2}}AW^{-1}K\|_2$ with respect to $\|\cdot\|_w$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$.

Proof: The proof is similar to Theorem 6. Define $J_w = \frac{ew^T}{\sum_i w_i}$ and $B = A - J_w$. Note that $J_w W^{-1}K = 0$. Then

$$\begin{aligned} \|W^{\frac{1}{2}}AW^{-1}K\|_2 &= \|W^{\frac{1}{2}}BW^{-1}K\|_2 \\ &= \max_{\|Kx\|_2=1} \|W^{\frac{1}{2}}BW^{-1}Kx\|_2 \\ &= \max_{x \perp e, \|x\|_2=1} \|W^{\frac{1}{2}}BW^{-1}x\|_2 \end{aligned}$$

Now $x \perp e$ if and only if $W^{-1}x \perp w$. Since $\|x\|_2 = \|W^{-\frac{1}{2}}x\|_w$, this means that $\|W^{\frac{1}{2}}AW^{-1}K\|_2 = \max_{x \perp w, \|W^{\frac{1}{2}}x\|_w=1} \|W^{\frac{1}{2}}Bx\|_2$. Since $\max_i w_i = 1$, this means that $\|W^{\frac{1}{2}}x\|_w = \sqrt{\sum_i (w_i x_i)^2} \leq \|x\|_w$ and thus

$$\|W^{\frac{1}{2}}AW^{-1}K\|_2 \geq \max_{x \perp w, \|x\|_w=1} \|W^{\frac{1}{2}}Bx\|_2$$

It is straightforward to show that $P(x) = J_w x$ and thus $d(Ax, X^*) = \|Bx\|_w = \|W^{\frac{1}{2}}Bx\|_2$. Since A has constant row sums, $A(X^*) \subseteq X^*$ and by Lemma 3 $c(A) = \max_{P(x)=0, \|x\|_w=1} d(Ax, X^*) = \max_{P(x)=0, \|x\|_w=1} \|W^{\frac{1}{2}}Bx\|_2$. Since $P(x) = 0$ if and only if $x \perp w$, this means that $c(A) \leq \|W^{\frac{1}{2}}AW^{-1}K\|_2$. \square

Note that the matrix A in Theorem 5, Theorem 6 and Theorem 7 is not necessarily nonnegative or stochastic.

3.4 examples

The matrix

$$A_1 = \begin{pmatrix} 1.1 & 0.0 & 0.0 \\ 0.6 & 0.5 & 0 \\ 0.6 & 0 & 0.5 \end{pmatrix}$$

is set-contracting with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since $\mu(A_1) = 0.6$ and $c(A_1) = 1.1 - \mu(A_1) = 0.5 < 1$. It is not pseudocontracting with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since $\|A_1\|_\infty = 1.1 > 1$.

The stochastic matrix

$$A_2 = \begin{pmatrix} 0.4 & 0.3 & 0.3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is set-nonexpanding with respect to $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since $\|A_2 K\|_2 = 1$ but it is not nonexpanding with respect to $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since $\|A_2\|_2 > 1$. Furthermore, Theorem 4 shows that A_2 is not set-contractive with respect to any weakly monotone norm and X^* .

The stochastic matrix

$$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

is set-contractive with respect to $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since $\|A_3 K\|_2 = 0.939$. Since $\|A_3\|_2 > 1$ it is not nonexpanding nor pseudocontractive with respect to $\|\cdot\|_2$ and X^* . It is also not pseudocontractive with respect to $\|\cdot\|_\infty$ and X^* since it is not scrambling.

The stochastic matrix

$$A_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0.9 & 0.1 & 0 \\ 0.1 & 0.1 & 0.8 \end{pmatrix}$$

has an interaction digraph that contains a spanning directed tree. However, it is not set-nonexpanding with respect to $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since $\|A_4 K\|_2 = 1.125 > 1$. This shows that the converse of Theorem 4 is not true for $\|\cdot\|_2$.⁴ On the other hand, A_4 is set-contractive with respect to $\|\cdot\|_\infty$ and

⁴ Theorem 3 shows that the converse of Theorem 4 is false as well for stochastic matrices with respect to $\|\cdot\|_\infty$ and X^* .

X^* since A_4 is a scrambling matrix. Furthermore, A_4 is set-contractive with respect to $\|\cdot\|_w$ and X^* for $w = (1, 0.2265, 1)^T$ since $\|W^{\frac{1}{2}}A_4W^{-1}K\|_2 < 1$.

Next we show some convergence results for dynamical systems of the form $x(k+1) = T_k x(k)$ where some T_k 's are set-contractive operators.

Theorem 8 *Let $\{T_k\}$ be a sequence of set-nonexpansive operators with respect to $\|\cdot\|$ and X^* and suppose that*

$$\lim_{k \rightarrow \infty} \prod_k c(T_k) = 0$$

Let $x(k+1) = T_k x(k)$. For any initial vector $x(0)$, $\lim_{k \rightarrow \infty} d(x(k), X^) = 0$.*

Proof: From Lemma 2, $c(\prod_k T_k) \leq \prod_k c(T_k) \rightarrow 0$ as $k \rightarrow \infty$ and the conclusion follows. \square

Theorem 9 *Let $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ and $\{A_k\}$ be a sequence of n by n constant row sum nonnegative matrices such that*

- *the diagonal elements are positive;*
- *all nonzero elements are equal to or larger than ϵ ;*
- *the row sum is equal to or less than r .*

If $r^{n-1} - \epsilon^{n-1} < 1$ and for each k , the interaction digraph of A_k contains a spanning directed tree, then $\lim_{k \rightarrow \infty} d(x(k), X^) = 0$ where $x(k+1) = A_k x(k)$.*

Proof: As discussed above, products of $n-1$ matrices A_k is scrambling. By definition, since each A_k has nonzero elements equal to or larger than ϵ , the nonzero elements of this product, denoted as P , will be equal to or larger than ϵ^{n-1} . This means that $\mu(P) \geq \epsilon^{n-1}$ and thus $\delta(P) \leq r^{n-1} - \epsilon^{n-1} < 1$ since P has row sums $\leq r^{n-1}$. Therefore P is set-contractive with respect to $\|\cdot\|_\infty$ and X^* with $c(P) \leq r^{n-1} - \epsilon^{n-1} < 1$. The result then follows from Theorem 8. \square

The following result shows existence of linear operators B_k and vectors $x_k^* \in X^*$ such that $x(k+1) = B_k x(k) + x_k^*$ has the same dynamics as $x(k+1) = T_k x(k)$. In particular, for $y(k+1) = B_k y(k)$ and $x(k+1) = T_k x(k)$, $d(y(k), X^*) = d(x(k), X^*)$ for all k .

Theorem 10 *T is a set-nonexpansive operator with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ if and only if for each $x \in \mathbb{R}^n$ there exists a stochastic matrix B and a vector $x^* \in X^*$ such that $T(x) = Bx + x^*$.*

T is a set-contractive operator with respect to $\|\cdot\|_\infty$ and $X^ = \{\alpha e : \alpha \in \mathbb{R}\}$ if and only if for each $x \in \mathbb{R}^n$ there exists a scrambling stochastic matrix B and a vector $x^* \in X^*$ such that $T(x) = Bx + x^*$.*

Proof: One direction of both statements follows from Theorem 3. Suppose T is set-nonexpansive and fix $x \in \mathbb{R}^n$. Define $x^* = P(T(x)) - P(x)$ which is a vector in X^* . Let $y = T(x) - x^*$. Then $P(y) = P(T(x)) - x^* = P(x)$ and by Lemma 1,

$$\min_i x_i \leq \min_i y_i \leq \max_i y_i \leq \max_i x_i$$

and thus there exists a stochastic matrix B such that $Bx = y$.

If T is set-contractive, then for $x \in X^*$, we can choose $B = \frac{1}{n}ee^T$ and $T(x) - Bx \in X^*$. For $x \notin X^*$, $d(x, X^*) < d(T(x), X^*)$. Define x^* and y as before and we see that

$$\min_i x_i < \min_i y_i \leq \max_i y_i < \max_i x_i$$

If $x_{i'} = \min_i x$, then it is clear that we can pick B with $Bx = y$ such that the i' -th column of B is positive, i.e. B is scrambling. \square

It can be beneficial to consider set-contractivity with respect to different norms. For instance, consider $x(k+1) = A_k x(k)$ where A_k are matrices that are not pseudocontractive with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ and whose diagonal elements are 0. Since the diagonal elements are not positive, the techniques in [3] cannot be used to show that products of A_k are pseudocontractive with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$. However, it is possible that A_k are set-contractive with respect to a different norm and thus convergence of $x(k)$ can be obtained by studying set-contractivity using this norm. For instance, the stochastic matrix

$$A = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{pmatrix}$$

has zeros on the diagonal and is not pseudocontractive with respect to $\|\cdot\|_\infty$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since A is not scrambling. On the other hand, A is set-contractive with respect to $\|\cdot\|_2$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ since $\|AK\|_2 = 0.939 < 1$.

For a set of constant row sum matrices A_k and $x(k+1) = A_k x(k)$, a lower bound for the exponential rate at which $x(k)$ approach $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ is $-\ln(c(A))$. The above examples show that there are matrices for which this rate is 0 for $\|\cdot\|_\infty$ and positive for $\|\cdot\|_2$ and other matrices for which the rate is positive and 0 for $\|\cdot\|_\infty$ and $\|\cdot\|_2$ respectively.

On the other hand, even though set-contractivity depends on the norm used, the equivalence of norms on \mathbb{R}^n and Lemma 4 provides the following result.

Theorem 11 *Let X^* be a closed set such that $\alpha X^* \subseteq X^*$ for all $\alpha \in \mathbb{R}$. and let H be a compact set of set-contractive matrices with respect to $\|\cdot\|_p$ and X^* .*

Then there exists m such that a product of m matrices in H is set-contractive with respect to $\|\cdot\|_q$.

Corollary 1 *Let H be a compact set of stochastic set-contractive matrices with respect to $\|\cdot\|_p$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$. Then a sufficiently long product of matrices in H is scrambling.*

4 Weak ergodicity of inhomogeneous Markov chains

In Section 2 we noted the connection between set-contractivity with respect to $\|\cdot\|_\infty$ and weak ergodicity in inhomogeneous Markov chains. In this section we elaborate on this connection. A sequence of stochastic matrices A_i is *weakly ergodic* if for each r , $\delta(A_r A_{r+1} \cdots A_{r+k}) \rightarrow 0$ as $k \rightarrow \infty$.

In [11] a *coefficient of ergodicity* is defined as a continuous function μ on the set of n by n stochastic matrices such that $0 \leq \mu(A) \leq 1$. A coefficient of ergodicity μ is *proper* if

$$\mu(A) = 1 \Leftrightarrow A = ev^T \quad \text{for some probability vector } v.$$

Seneta [11] gives the following necessary and sufficient conditions for weak ergodicity generalizing the arguments by Hajnal.

Theorem 12 *Suppose μ_1 and μ_2 are coefficients of ergodicity such that μ_1 is proper and the following equation is satisfied for some constant C and all k ,*

$$1 - \mu_1(S_1 S_2 \cdots S_k) \leq C \prod_{i=1}^k (1 - \mu_2(S_i)) \quad (7)$$

where S_i are stochastic matrices. Then a sequence of stochastic matrices A_i is weakly ergodic if there exists a strictly increasing subsequence $\{i_j\}$ such that

$$\sum_{j=1}^{\infty} \mu_2(A_{i_j+1} \cdots A_{i_{j+1}}) = \infty \quad (8)$$

Conversely, if A_i is a weakly ergodic sequence, and μ_1, μ_2 are both proper coefficients of ergodicity satisfying Eq. (7), then Eq. (8) is satisfied for some strictly increasing sequence $\{i_j\}$.

Define H as the set of stochastic matrices that are set-nonexpansive with respect to a norm $\|\cdot\|$ and $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$. For $\|\cdot\|_\infty$, H is the set of stochastic matrices. Let us define $\mu_c(A) = 1 - c(A)$. Then μ_c is a proper coefficient of ergodicity when restricted to H . This can be seen as follows.

Clearly $0 \leq \mu_c(A) \leq 1$. If $A = ev^T$, then $Ax \in X^*$ and thus $c(A) = 0$ and $\mu_c(A) = 1$. If $A \neq ev^T$, then there exists i, j, k such that $A_{ik} \neq A_{jk}$. Let x be the k -th unit basis vector. Then $(Ax)_i \neq (Ax)_j$, i.e. $d(Ax, X^*) > 0$, $c(A) > 0$ and $\mu_c(A) < 1$. By choosing $\mu_1 = \mu_2 = \mu_c$, Eq. (7) is satisfied with $C = 1$ by Lemma 2. Thus we have shown that a sufficient and necessary condition for a sequence of matrices in H to be weakly ergodic is

$$\sum_{j=1}^{\infty} 1 - c(A_{i_j+1} \cdots A_{i_{j+1}}) = \infty$$

for some strictly increasing subsequence $\{i_j\}$.

5 Application to the synchronization of coupled map lattices

Coupled map lattices [12] have been studied extensively and have been shown to exhibit complex behavior [13,14]. Recently, synchronization in coupled map lattice has attracted considerable attention [15,16,17,18,19]. We show here how set-contractivity can be useful in studying synchronization in coupled map lattices.

Given a map $f_k : \mathbb{R} \rightarrow \mathbb{R}$, consider state variables $x_i \in \mathbb{R}$ which evolve according to f_k at time k : $x_i(k+1) = f_k(x_i(k))$. By coupling the output of these maps we obtain a coupled map lattice where each state evolves as:

$$x_i(k+1) = \sum_j a_{ij}(k) f_k(x_j(k))$$

This can be rewritten as

$$x(k+1) = A_k F_k(x(k)) \tag{9}$$

where $x(k) = (x_1(k), \dots, x_n(k))^T \in \mathbb{R}^n$ and $F_k(x(k)) = (f_k(x_1(k)), \dots, f_k(x_n(k)))^T$. We assume that A_k is a constant row sum matrix for all k . The map f_k depends on k , i.e. we allow the map in the lattice to be time varying. Furthermore, we do not require A_k to be a nonnegative matrix. We say the coupled map lattice in Eq. (9) *synchronizes* if $\lim_{k \rightarrow \infty} |x_i(k) - x_j(k)| = 0$ for all i and j , i.e. $x(k)$ approaches the synchronization manifold $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ as $k \rightarrow \infty$. If the row sum of A_k is 1, then this means that at synchronization, each state x_i in the lattice exhibits dynamics of the uncoupled map f_k , i.e. if $x(h) \in X^*$, then for all $k \geq h$, $x(k) \in X^*$ and $x_i(k+1) = f_k(x_i(k))$.

We are now ready to state our synchronization result:

Theorem 13 *Let ρ_k be the Lipschitz constant of f_k . If $\lim_{k \rightarrow \infty} \prod_k c(A_k) \rho_k = 0$, where $c(A_k)$ is the set-contractivity with respect to $X^* = \{\alpha e : \alpha \in \mathbb{R}\}$ and*

a monotone norm, then the coupled map lattice in Eq. (9) synchronizes.

Proof:

$$\|F_k(x(k)) - P(F_k(x(k)))\| \leq \|F_k(x(k)) - F_k(P(x(k)))\| \leq \rho_k \|x(k) - P(x(k))\|$$

where the last inequality follows from monotonicity of the norm. This implies that $c(F_k) \leq \rho_k$ and the result follows from Theorem 8. \square

Thus we can synchronize the coupled map lattice if we can find matrices A_k and a norm such that the contractivities $c(A_k)$ are small enough.

Corollary 2 *Let ρ_k be the Lipschitz constant of f_k . If $\sup_k r(A_K) - \mu(A_K) - \frac{1}{\rho_k} < 0$, then Eq. (9) synchronizes⁵.*

Proof: Follows by applying Theorem 13 to set-contractivity with respect to $\|\cdot\|_\infty$. \square

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⁵ Here $r(A)$ denotes the row sum of the matrix A .

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